Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Approximation Theory 139 (2006) 65–74

Journal of Approximation Theory

www.elsevier.com/locate/jat

Lieb–Thirring and Bargmann-type inequalities for circular arc

Leonid Golinskii

Mathematics Division, B. Verkin Institute for Low Temperature Physics and Engineering, 47 Lenin Avenue, 61103 Kharkov, Ukraine

Received 29 December 2004; accepted 25 April 2005

Communicated by Andrej Zlatos
Available online 28 July 2005

Abstract

For measures on the unit circle with convergent Verblunsky coefficients we find relations in form of inequalities between these coefficients and the distances from mass points to the essential support of the measure.

1. Introduction

Let μ be a nontrivial (i.e., not a finite combination of delta measures) probability measure on the unit circle $\mathbb{T}$. The orthonormal with respect to μ polynomials

$$\varphi_n(z) = \varphi_n(\mu, z) = \kappa_n(\mu) z^n + \cdots$$

are uniquely determined by the requirement that $\kappa_n(\mu) > 0$ and

$$\int_{\mathbb{T}} \varphi_n(\mu, \zeta)\overline{\varphi_m(\mu, \zeta)}\, d\mu(\zeta) = \delta_{n,m}, \qquad n, m \in \mathbb{Z}_+ = \{0, 1, \ldots\}, \quad \zeta \in \mathbb{T}.$$

The monic orthogonal polynomials are

$$\Phi_n(\mu, z) = \kappa_n^{-1} \varphi_n(\mu, z) = z^n + \cdots .$$

E-mail address: golinskii@ilt.kharkov.ua.

0021-9045/$ - see front matter © 2005 Elsevier Inc. All rights reserved.
doi:10.1016/j.jat.2005.04.014

The parameters $\alpha_n = -\overline{\Phi_{n+1}(0)}$, $\alpha_{-1} = -1$ are called *Verblunsky coefficients* after [10] where they appeared for the first time (in a different setting) They play a key role in the theory of orthogonal polynomials on the unit circle (OPUC) due to a fundamental result of Verblunsky that $\mu \leftrightarrow \{\alpha_n\}_{n=0}^{\infty}$ sets up a one-one correspondence between nontrivial measures on $\mathbb{T}$ and the direct product of unit disks $\otimes_{n=0}^{\infty} \mathbb{D}$, that is, each sequences $\{\alpha_n\}$ of complex numbers from the open unit disk $\mathbb{D}$ comes up as a sequence of Verblunsky coefficients for a certain uniquely determined nontrivial probability measure μ on the unit circle.

It was Geronimus who put these parameters in force in the theory of OPUC in early 40s and proved a number of remarkable results concerning properties of measures on $\mathbb{T}$ based on specific behavior of their Verblunsky coefficients (so they could equally well be named *Geronimus parameters*).

In a fundamental treatise [8,9] Simon suggests a new approach to the relation $\mu \leftrightarrow \{\alpha_n\}_{n=0}^{\infty}$ as a spectral theory problem analogous to the association of a potential V to the spectral measure of the corresponding Schrödinger operator or Jacobi parameters to a measure in the theory of orthogonal polynomials on the real line.

The problem discussed in the present note is inspired by the following results from [9, Section 12.2].

Theorem S1. *Suppose μ has Verblunsky coefficients α_j and $\beta_{j+m} = \beta_j$, $j \in \mathbb{Z}_+$, for some $m \geqslant 1$. If*

$$\sum_{j=0}^{\infty} |\alpha_j - \beta_j|^q < \infty$$

for some $q \geqslant 1$, then

$$\sum_{\zeta_j} \operatorname{dist}(\zeta_j,\ \operatorname{ess\,supp}(d\mu))^p < \infty,$$

where ζ_j are mass points in gaps, $p > \frac{1}{2}$ if $q = 1$ and $p \geqslant q - \frac{1}{2}$ if $q > 1$.

Theorem S2. *Suppose α's and β's are as in Theorem* S1. *If*

$$\sum_{j=0}^{\infty} j|\alpha_j - \beta_j| < \infty$$

then μ has an essential support whose complement has at most m gaps, and each gap has only finitely many mass points.

The first result is a bound of Lieb–Thirring type [7], while the second one is of Bargmann type [1].

In the present paper we study the simplest case $m = 1$ and so $\beta_0 = \beta_1 = \cdots = \alpha$ (measures and orthogonal polynomials on one arc). Our goal is to find quantitative results in form of inequalities from which the above theorems for almost constant Verblunsky coefficients drop out immediately. It turns out that Theorem S1 holds for $q \geqslant 1$ and $p \geqslant q - \frac{1}{2}$, as it belongs (cf. [5,6]).

It is well known (see [3, Theorem 1′], [8, Example 4.3.10]) that the support of a measure μ with convergent Verblunsky coefficients

$$\lim_{n\to\infty} \alpha_n = \alpha, \quad 0 < |\alpha| < 1, \tag{1.1}$$

is composed of a closed arc (essential support)

$$\Delta_\gamma = \{e^{it} : \gamma \leqslant t \leqslant 2\pi - \gamma\}, \qquad \sin\frac{\gamma}{2} = |\alpha|, \quad 0 < \gamma < \pi, \tag{1.2}$$

and a set of mass points $\zeta_n^\pm = e^{i\gamma_n^\pm}$ off this arc, where we put $\Im\zeta_n^+ \geqslant 0$, $\Im\zeta_n^- < 0$.

Theorem 1.1. *Suppose μ has Verblunsky coefficients α_n which satisfy* (1.1). *Then for the mass points $\zeta_n^\pm$*

$$\sum_n (|\zeta_n^+ - e^{i\gamma}|^p + |\zeta_n^- - e^{-i\gamma}|^p) \leqslant C(p, |\alpha|) \sum_{n=-1}^{\infty} |\alpha_n - \alpha|^{p+1/2}, \tag{1.3}$$

where $p \geqslant 1/2$ and a positive constant C depends on p and $|\alpha|$.

Theorem 1.2. *Suppose α_n's and α are as in Theorem* 1.1. *If*

$$\sum_{n=0}^{\infty} n|\alpha_n - \alpha| < \infty, \tag{1.4}$$

then the number $N(\mu)$ of mass points of μ off the essential support is bounded from above by

$$N(\mu) \leqslant C(|\alpha|) \sum_{n=-1}^{\infty} (n+2)|\alpha_n - \alpha|, \tag{1.5}$$

where a positive constant C depends only on $|\alpha|$.

It may look strange enough that summation in the RHS (1.3) and (1.5) is taken from $n = -1$ and so the expression on the right is always strictly positive. However, the simplest example of the Geronimus polynomials with $\alpha_n = \alpha$, $n = 0, 1, \ldots$ and $|\alpha + 1/2| > \frac{1}{2}$, for which the orthogonality measure has one mass point off the essential support (cf. [8, Theorem 1.6.13]), shows that there is no hope to have inequalities of the form (1.3) or (1.5) with the sum taken from $n = 0$.

There are two main ingredients of the proof. First, we apply Zhukovsky's transform rather than Caley's to the corresponding CMV matrix and work with symmetric five-diagonal matrices. Secondly, we make up a Jacobi-type model (see Section 2) which relates five- and three-diagonal matrices and enables one to use the results for Jacobi matrices to the full extent. Theorems 1.1 and 1.2 are proved is Sections 3 and 4, respectively.

It is my pleasure to wish Barry Simon, now the world renowned expert in orthogonal polynomials on the unit circle, much success and many more years of creative life.

2. Jacobi-type model for five-diagonal matrices

Given a Hilbert space H, take the orthogonal sum of two identical copies of it

$$H \oplus H = \{(h', h''); \ h', h'' \in H\}$$

with the inner product

$$\langle (h', h''), (g', g'') \rangle = \langle h', g' \rangle + \langle h'', g'' \rangle.$$

For an orthonormal basis $\{e_k\}_{k \geqslant 0}$ in H define

$$\tilde{e}_{2k} := (e_k, 0), \qquad \tilde{e}_{2k+1} := (0, e_k), \quad k \in \mathbb{Z}_+,$$

and so $\{\tilde{e}_k\}_{k \geqslant 0}$ is the orthonormal basis in $H \oplus H$.

There is a canonical isomorphism $U : H \to H \oplus H$, given by $Ue_j = \tilde{e}_j$, which maps any vector $h = \sum_{k \geqslant 0} h_k e_k$ to

$$Uh = (h', h''), \quad h' = \sum_{k \geqslant 0} h_{2k} e_k, \quad h'' = \sum_{k \geqslant 0} h_{2k+1} e_k.$$

Let $A = \|a_{ij}\|_0^\infty$ be a bounded linear operator in H given by its matrix in the basis $\{e_k\}$. It is clear that A is unitarily equivalent to its model in $H \oplus H$

$$\widetilde{A} = UAU^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where

$$\begin{aligned} &A_{11} = \|a_{2i,2j}\|, \quad A_{12} = \|a_{2i,2j+1}\|, \quad A_{21} = \|a_{2i+1,2j}\|, \\ &A_{22} = \|a_{2i+1,2j+1}\|. \end{aligned} \tag{2.1}$$

Conversely, if an operator

$$\widetilde{A} = \begin{bmatrix} \widetilde{P} & \widetilde{Q} \\ \widetilde{R} & \widetilde{S} \end{bmatrix}$$

acts in $H \oplus H$, then $A = U^{-1}\widetilde{A}U = \|a_{ij}\|_0^\infty$ with

$$a_{2i,2j} = \tilde{p}_{ij}, \quad a_{2i,2j+1} = \tilde{q}_{ij}, \quad a_{2i+1,2j} = \tilde{r}_{ij}, \quad a_{2i+1,2j+1} = \tilde{s}_{ij}.$$

In what follows we shall deal with symmetric five-diagonal matrices (operators in $\ell^2(\mathbb{Z}_+)$) of the form

$$F = \begin{pmatrix} r_0 & \bar{q}_0 & p_0 & & & \\ q_0 & r_1 & \bar{q}_1 & p_1 & & \\ p_0 & q_1 & r_2 & \bar{q}_2 & p_2 & \\ & p_1 & q_2 & r_3 & \bar{q}_3 & p_3 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix} = F(\{p_n\}, \{q_n\}, \{r_n\}),$$

$p_i > 0, r_i = \bar{r}_i$. The notation here is consistent with the standard one for Jacobi matrices $J(\{a_n\}, \{b_n\})$ with the main diagonal $\{b_n\}$ and the second diagonal $\{a_n\}$. The bi-diagonal matrices

$$\begin{pmatrix} v_0 & & & & \\ u_0 & v_1 & & & \\ & u_1 & v_2 & & \\ & & u_2 & v_3 & \\ & & & \ddots & \ddots \end{pmatrix} = D(\{u_n\}, \{v_n\})$$

will also appear on the scene. The matrix $\widetilde{F} = UFU^{-1}$ is called a *Jacobi-type model* for F. According to the general formula (2.1)

$$\widetilde{F} = \begin{bmatrix} J_{11} & D \\ D^* & J_{22} \end{bmatrix} \tag{2.2}$$

with

$$J_{11} = J(\{p_{2k}\}, \{r_{2k}\}), \quad J_{22} = J(\{p_{2k+1}\}, \{r_{2k+1}\}), \quad D = D(\{q_{2k+1}\}, \{\bar{q}_{2k}\}),$$

$k \in \mathbb{Z}_+$. In particular, if $q_i = 0$, then F is unitarily equivalent to the orthogonal sum of two Jacobi matrices.

Let μ be a probability measure on the unit circle $\mathbb{T}$ with Verblunsky coefficients $\{\alpha_n(\mu)\}_0^\infty$, $\alpha_{-1} = -1$ and CMV matrix $\mathcal{C}(\mu)$ (see [2], [8, Section 4.2]). The main object we are working with is the five-diagonal matrix

$$F = F(\{p_n\}, \{q_n\}, \{r_n\}) = \mathcal{C}(\mu) + \mathcal{C}^*(\mu) = \mathcal{C}(\mu) + \mathcal{C}^{-1}(\mu). \tag{2.3}$$

It is easily seen from the expression for CMV matrices ([2], [8, Proposition 4.2.3]) that

$$r_k = -2\Re\overline{\alpha}_{k-1}\alpha_k, \quad p_k = \rho_k\rho_{k+1}, \quad q_{2k} = \rho_{2k}\delta_{2k}, \quad q_{2k+1} = \rho_{2k+1}\overline{\delta}_{2k+1}, \tag{2.4}$$

$k \in \mathbb{Z}_+$, where $\rho_n^2 := 1 - |\alpha_n|^2$, $\delta_n := \alpha_{n+1} - \alpha_{n-1}$.

Example. Let $\alpha_n \equiv \alpha$, $|\alpha| < 1$. Then

$$r_0 = 2\Re\alpha, \quad q_0 = \sqrt{1-|\alpha|^2}(\alpha+1),$$

$$r_1 = r_2 = \cdots = -2|\alpha|^2, \quad q_1 = q_2 = \cdots = 0,$$
$$p_0 = p_1 = \cdots = 1 - |\alpha|^2 = \rho^2.$$

So $D = \operatorname{diag}\{\rho(\alpha+1), 0, 0, \ldots\}$,

$$J_{11} = \begin{pmatrix} 2\Re\alpha & \rho^2 & & & \\ \rho^2 & -2|\alpha|^2 & \rho^2 & & \\ & \rho^2 & -2|\alpha|^2 & \rho^2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

$$J_{22} = \begin{pmatrix} -2|\alpha|^2 & \rho^2 & & & \\ \rho^2 & -2|\alpha|^2 & \rho^2 & & \\ & \rho^2 & -2|\alpha|^2 & \rho^2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

The above example suggests that in the case $\lim_n \alpha_n = \alpha$ we are interested in the right scaling for the Jacobi-type model $\widetilde{F}$ would be

$$\widehat{F} = \frac{1}{1-|\alpha|^2}\,(\widetilde{F} + 2|\alpha|^2) = \begin{bmatrix} \widehat{J}_{11} & \widehat{D} \\ \widehat{D}^* & \widehat{J}_{22} \end{bmatrix}, \tag{2.5}$$

$$\widehat{J}_{11} = J\left(\left\{\frac{-2\Re\bar{\alpha}_{2k-1}\alpha_{2k} + 2|\alpha|^2}{1-|\alpha|^2}\right\}, \left\{\frac{\rho_{2k}\rho_{2k+1}}{1-|\alpha|^2}\right\}\right), \tag{2.6}$$

$$\widehat{J}_{22} = J\left(\left\{\frac{-2\Re\bar{\alpha}_{2k}\alpha_{2k+1} + 2|\alpha|^2}{1-|\alpha|^2}\right\}, \left\{\frac{\rho_{2k+1}\rho_{2k+2}}{1-|\alpha|^2}\right\}\right), \tag{2.7}$$

$$\widehat{D} = D\left(\left\{\frac{\rho_{2k}\bar{\delta}_{2k}}{1-|\alpha|^2}\right\}, \left\{\frac{\rho_{2k+1}\bar{\delta}_{2k+1}}{1-|\alpha|^2}\right\}\right). \tag{2.8}$$

It follows directly from (2.5) to (2.8) that under the assumption $\lim_n \alpha_n = \alpha$, $|\alpha| < 1$, the scaled Jacobi-type model $\widehat{F}$ satisfies

$$\widehat{F} = \begin{bmatrix} J_0 & 0 \\ 0 & J_0 \end{bmatrix} + K,$$

where $J_0 = J(\{1\}, \{0\})$ is the free Jacobi matrix and K a compact operator.

3. Lieb–Thirring inequalities

As usual, we denote by $\sigma(T)$ the spectrum of a bounded linear operator T. It is well known that under assumption (1.1) the spectrum of the corresponding CMV matrix $\mathcal{C}(\mu)$ is

$$\sigma(\mathcal{C}(\mu)) = \Delta_\gamma \cup \{e^{i\gamma_n^\pm}\},$$

where Δ_γ is the closed arc (1.2) and $\{e^{i\gamma_n^+}\}$ ($\{e^{i\gamma_n^-}\}$) a discrete set of eigenvalues on the upper (lower) semicircle, off the arc Δ_γ, with the endpoints $e^{i\gamma}$ ($e^{-i\gamma}$) being the only possible accumulation points. By the Spectral Mapping Theorem for the spectrum of $\widehat{F}$ we have

$$\sigma(\widehat{F}) = [-2, 2] \bigcup \{\tau_n^\pm\}, \quad \tau_n^\pm = \frac{2\cos\gamma_n^\pm + 2|\alpha|^2}{1-|\alpha|^2} \searrow 2 \tag{3.1}$$

as $n \to \infty$. Clearly,

$$\begin{aligned} \tau_n^+ - 2 &= \frac{2}{1-|\alpha|^2}(\cos\gamma_n^+ - \cos\gamma) = \frac{4}{1-|\alpha|^2}\,\sin\frac{\gamma_n^+ + \gamma}{2}\,\sin\frac{\gamma_n^+ - \gamma}{2} \\ &= \frac{2}{1-|\alpha|^2}\,\sin\frac{\gamma_n^+ + \gamma}{2}\left|e^{i\gamma_n^+} - e^{i\gamma}\right| \end{aligned}$$

$$\geqslant \sin\,\frac{\gamma_n^+ + \gamma}{2}\,\min\left(\sin\,\frac{\gamma}{2}, \sin\gamma\right)|e^{i\gamma_n^+} - e^{i\gamma}|$$

and so

$$\frac{1-|\alpha|^2}{2}(\tau_n^+ - 2) \leqslant |e^{i\gamma_n^+} - e^{i\gamma}| \leqslant \frac{1-|\alpha|^2}{2\min\left(\sin\frac{\gamma}{2}, \sin\gamma\right)}(\tau_n^+ - 2).$$

The similar bounds hold for $\tau_n^- - 2$ and $|e^{i\gamma_n^-} - e^{-i\gamma}|$. It will be advisable to have a single sequence $\{\tau_n\}$ obtained from $\{\tau_n^{\pm}\}$ by proper reordering so $\tau_1 \geqslant \tau_2 \geqslant \cdots \to 2$. Thus, for every $p > 0$

$$K_1 \sum_n (\tau_n - 2)^p \leqslant \sum_n (|e^{i\gamma_n^+} - e^{i\gamma}|^p + |\,e^{i\gamma_n^-} - e^{-i\gamma}|^p) \leqslant K_2 \sum_n (\tau_n - 2)^p \quad (3.2)$$

with positive constants K_1, K_2 which depend on $|\alpha|$ and p.

Our next step is to find a block Jacobi matrix which dominates the scaled Jacobi-type model $\widehat{F}$ (2.5). Let

$$G = \begin{bmatrix} G_{11} & D \\ D^* & G_{22} \end{bmatrix} \quad (3.3)$$

with

$$G_{11} = J(\{p_{2k}\}, \{r_{2k}\}), \quad G_{22} = J(\{p_{2k+1}\}, \{r_{2k+1}\}), \quad D = D(\{q_{2k+1}\}, \{\overline{q}_{2k}\}).$$

The calculation of its quadratic form gives

$$\langle G(h, g), (h, g)\rangle = \langle G_{11}h, h\rangle + \langle G_{22}g, g\rangle + 2\Re\langle Dg, h\rangle,$$
$$\langle Dg, h\rangle = \sum_{k=0}^{\infty}(q_{2k-1}g_{k-1} + \overline{q}_{2k}g_k)\overline{h}_k, \quad q_{-1} = 0$$

and

$$2|\langle Dg, h\rangle| \leqslant \sum_{k=0}^{\infty}(|q_{2k-1}| + |q_{2k}|)|h_k|^2 + \sum_{k=0}^{\infty}(|q_{2k}| + |q_{2k+1}|)|g_k|^2.$$

Now put

$$G'_{11} := G_{11} + \mathrm{diag}\{|q_{2n-1}| + |q_{2n}|\}, \quad G'_{22} := G_{22} + \mathrm{diag}\{|q_{2n}| + |q_{2n+1}|\}$$

and so the desired matrix takes the form

$$G' = \begin{bmatrix} G'_{11} & 0 \\ 0 & G'_{22} \end{bmatrix}, \quad G' \geqslant G. \quad (3.4)$$

We apply the above procedure to $G = \widehat{F}$ (2.5):

$$\widehat{F} \leqslant F' = \begin{bmatrix} J'_{11} & 0 \\ 0 & J'_{22} \end{bmatrix}, \quad J'_{ii} = J(\{a_{\mathrm{in}}\}, \{b_{\mathrm{in}}\}), \; i = 1, 2 \quad (3.5)$$

with

$$b_{1n}=\frac{-2\Re\overline{\alpha}_{2n-1}\alpha_{2n}+2|\alpha|^2}{1-|\alpha|^2}+|q_{2n-1}|+|q_{2n}|,\quad a_{1n}=\frac{\rho_{2n}\rho_{2n+1}}{1-|\alpha|^2},\tag{3.6}$$

$$b_{2n}=\frac{-2\Re\overline{\alpha}_{2n}\alpha_{2n+1}+2|\alpha|^2}{1-|\alpha|^2}+|q_{2n}|+|q_{2n+1}|,\quad a_{2n}=\frac{\rho_{2n+1}\rho_{2n+2}}{1-|\alpha|^2}\tag{3.7}$$

and $|q_m|=(1-|\alpha|^2)^{-1}\rho_m|\delta_m|$ (see (2.6)–(2.8)). It is clear from the convergence assumption that $J'_{ii}=J_0+K_i$ with compact operators K_i, $i=1,2$.

The spectrum of the RHS in (3.5) is under control:

$$\sigma(F')=\sigma(J'_{11})\bigcup\sigma(J'_{22}),$$

$$\sigma(J'_{ii})=[-2,2]\bigcup\{\lambda_{\rm in}\}_{n\geqslant 1},\qquad \lambda_{i1}>\lambda_{i2}>\cdots\to 2,\quad i=1,2$$

(there is no spectrum below -2 by (3.1) and (3.5)). Moreover,

$$\sum_n f(\tau_n-2)\leqslant\sum_{i=1}^{2}\sum_n f(\lambda_{\rm in}-2)\tag{3.8}$$

for each nonnegative and nondecreasing function f on $(0,\infty)$. Put $f(x)=x^p$, $p\geqslant\frac{1}{2}$. The LHS in (3.7) is bounded from below thanks to (3.2):

$$\sum_n(\tau_n-2)^p\geqslant K_3\sum_n(|e^{i\gamma_n^+}-e^{i\gamma}|^p+|e^{i\gamma_n^-}-e^{-i\gamma}|^p).$$

As for the RHS in (3.7) we apply the standard Lieb–Thirring inequality for Jacobi matrices [6, Theorem 2] to obtain

$$\sum_{i=1}^{2}\sum_n(\lambda_{\rm in}-2)^p\leqslant\kappa(p)\left\{\sum_{i=1}^{2}\sum_n(|b_{\rm in}|^{p+1/2}+|a_{\rm in}-1|^{p+1/2})\right\}.$$

It remains only to go over to the Verblunsky coefficients α_n by using (3.6), (3.7). The routine calculation shows that

$$|q_m|=\frac{\rho_m|\delta_m|}{1-|\alpha|^2}\leqslant\frac{|\alpha_{m+1}-\alpha|+|\alpha_{m-1}-\alpha|}{1-|\alpha|^2},$$

$$|\Re(\overline{\alpha}_{m-1}\alpha_m)-|\alpha|^2|\leqslant|\alpha_{m-1}-\alpha|+|\alpha_m-\alpha|,$$

$m\geqslant 0$, and hence

$$|b_{1n}|\leqslant\frac{3}{1-|\alpha|^2}(|\alpha_{2n-2}-\alpha|+\cdots+|\alpha_{2n+1}-\alpha|),$$

$$|b_{2n}|\leqslant\frac{3}{1-|\alpha|^2}(|\alpha_{2n-1}-\alpha|+\cdots+|\alpha_{2n+2}-\alpha|),$$

$$|a_{1n} - 1| \leqslant \frac{2}{(1-|\alpha|^2)^2}(|\alpha_{2n} - \alpha| + |\alpha_{2n+1} - \alpha|),$$

$$|a_{2n} - 1| \leqslant \frac{2}{(1-|\alpha|^2)^2}(|\alpha_{2n+1} - \alpha| + |\alpha_{2n+2} - \alpha|).$$

The combination of the above inequalities gives

$$\sum_n (|e^{i\gamma_n^+} - e^{i\gamma}|^p + |e^{i\gamma_n^-} - e^{-i\gamma}|^p) \leqslant C(p, |\alpha|) \sum_{n=-1}^{\infty} |\alpha_n - \alpha|^{p+1/2}, \tag{3.9}$$

which completes the proof of Theorem 1.1.

Remark. The main result can be proved under the weaker assumption

$$\lim_n |\alpha_n| = |\alpha|, \quad 0 < |\alpha| < 1; \qquad \lim_n \frac{\alpha_{n-1}}{\alpha_n} = 1,$$

known as the *López condition*. Inequality (3.9) takes the form

$$\sum_n (|e^{i\gamma_n^+} - e^{i\gamma}|^p + |e^{i\gamma_n^-} - e^{-i\gamma}|^p)$$
$$\leqslant C(p, |\alpha|) \sum_{n=-1}^{\infty} \left(||\alpha_n| - |\alpha||^p + \left| \frac{\alpha_{n-1}}{\alpha_n} - 1 \right|^p + \left| \frac{\alpha_n}{\alpha_{n-1}} - 1 \right|^p \right).$$

4. Bargmann-type bounds

Assume that the Verblunsky coefficients α_n satisfy

$$\sum_n n|\alpha_n - \alpha| < \infty, \quad 0 < |\alpha| < 1.$$

It was proved in [4, Theorem 12] that the number $N(\mu) = N(\mathcal{C})$ of eigenvalues $\{e^{i\gamma_n^\pm}\}$ of the CMV operator $\mathcal{C}(\mu)$ (the number of mass points of the orthogonality measure μ) off the arc Δ_γ (1.2) is finite. We are looking for quantitative bounds for the value $N(\mu)$ in terms of α_n.

The procedure described in the previous sections works perfectly well to solve this problem. It follows easily from the above construction that

$$N(\mathcal{C}) \leqslant 2N(\widehat{F}) \leqslant 2N(F'),$$

where $N(\widehat{F})$ ($N(F')$) is the number of eigenvalues of $\widehat{F}$ (F') outside $[-2, 2]$. Clearly, $N(F') \leqslant N(J'_{11}) + N(J'_{22})$ and to each term in the right-hand side standard Bargmann-type bound for Jacobi matrices [6, Theorem A.1] applies

$$N(J'_{ii}) \leqslant \sum_n (n|b_{\text{in}}| + (4n+2)|a_{\text{in}} - 1|), \quad i = 1, 2.$$

To complete the proof of Theorem 1.2 it remains only to go over from the Jacobi coefficients $b_{\text{in}}, a_{\text{in}}$ to Verblunsky coefficients α_n.

Acknowledgments

The work was partially supported by INTAS Research Network NeCCA 03-51-6637 and NATO Collaborative linkage Grant PST. CLG. 979738.

References

[1] V. Bargmann, On the number of bound states in a certain field of force, Proc. Nat. Acad. Sci. USA 38 (1952) 961–966.
[2] M.J. Cantero, L. Moral, L. Velázquez, Five-diagonal matrices and zeros of orthogonal polynomials on the unit circle, Linear Algebra Appl. 362 (2003) 29–56.
[3] Ya.L. Geronimus, On the character of the solution of the moment problem in the case of limit-periodic associated fraction, Izv. Akad. Nauk SSSR Ser. Mat. 5 (1941) 203–210 (in Russian).
[4] L. Golinskii, Operator theoretic approach to orthogonal polynomials on the unit circle, Math. Phys. Anal. Geom. 7 (2000) 3–34.
[5] D. Hundertmark, E.H. Lieb, L.E. Tomas, A sharp bound for an eigenvalue moment of the one-dimensional Schrödinger operator, Adv. Theor. Math. Phys. 2 (1998) 719–731.
[6] D. Hundertmark, B. Simon, Lieb–Thirring inequalities for Jacobi matrices, J. Approx. Theory 118 (2002) 106–130.
[7] E.H. Lieb, Bounds on the eigenvalues of the Laplace and Schrödinger operators, Bull. Amer. Math. Soc. 82 (1976) 751–753.
[8] B. Simon, Orthogonal Polynomials on the Unit Circle, vol. 1, AMS Colloquium Series, American Mathematical Society, Providence, RI, 2004.
[9] B. Simon, Orthogonal Polynomials on the Unit Circle, vol. 2, AMS Colloquium Series, American Mathematical Society, Providence, RI, 2004.
[10] S. Verblunsky, On positive harmonic functions: a contribution to the algebra of Fourier series, Proc. London Math. Soc. 38 (1935) 125–157.